Functional analysis for
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Definition 1. Let X be a linear space over a field K of real or complex numbers, that is $K = \mathbb{R}$ ora $K = \mathbb{C}$. We say that a function $\|\cdot\| : X \to [0, \infty)$ is a norm if the following conditions are satisifed

(N1) $\forall_{x \in X} \qquad \|x\| = 0 \iff x = 0$ *(nondegeneracy)* ,

(N2) $\forall_{x \in X}\ \forall_{\lambda \in K} \qquad \|\lambda x\| = |\lambda| \|x\|$ *(homogenity)* ,

(N3) $\forall_{x,y \in X} \qquad \|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality).*

Ex 2. Let $(X, \|\cdot\|)$ be a norm space and put $\rho(x, y) = \|x - y\|$, $x, y \in X$. Check that (X, ρ) is a metric space.

Ex 3. Show that for $p \geq 1$ the function $\|x\|_p = (\sum_{k=1}^{n} |x(k)|^p)^{\frac{1}{p}}$ is a norm in $\mathbb{R}^n$.

Ex 4. Examin the limit $\lim_{p \to \infty} \|x\|_p$, $x \in \mathbb{R}^n$, and show that $\|x\|_\infty = \max_{1 \leq k \leq n} |x(k)|$ is a norm in $\mathbb{R}^n$.

Ex 5. Check that spaces of *p-summable sequences*

$$l_p = \{x = (x(1), x(2), ...) : \|x\|_p = \left(\sum_{k=1}^{n} |x(k)|^p \right)^{\frac{1}{p}} < \infty\}, \qquad 1 \leq p < \infty,$$

are normed linear spaces (with $\|\cdot\|_p$ as a norm).

Ex 6. Prove that the space of *bounded sequences*

$$l_\infty = \{x = (x(1), x(2), ...) : \|x\|_\infty = \sup_{k \in \mathbb{N}} |x(k)| < \infty\},$$

the space of *convergent sequences*

$$c = \{x = (x(1), x(2), ...) : \exists_g \lim_{k \to \infty} x(k) = g\},$$

and the space of *sequences convergent to zero*

$$c_0 = \{x = (x(1), x(2), ...) : \lim_{k \to \infty} x(k) = 0\}$$

are normed linear spaces with $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x(k)|$ as a norm.

Ex 7. Prove that the space of *continous functions* $C[a, b]$ on the interval $[a, b]$ is a normed space with $\|x\|_\infty = \sup_{t \in [a,b]} |x(t)|$ as a norm.

Ex 8. Prove that the space of *continously differentiable functions* $C^{(1)}[a, b]$ on the interval $[a, b]$ is a normed space with $\|x\|_1 = \sup_{t \in [a,b]} |x(t)| + \sup_{t \in [a,b]} |x'(t)|$ as a norm.

Ex 9. Check whether a pair $(X, \|\cdot\|)$ is a norm space. If not check if (X, ρ) where $\rho(x, y) = \|x - y\|$ is metric space.

N	X	$\|\cdot\|$
1.	$\{x \in l_\infty : \sum_{k=1}^{\infty} \|x(k)\|^3 < 1\}$	$\|x\| = \sup_{k\in\mathbb{N}} \|x(k)\|$
2.	$\{x(t) \in C^{(1)}[0,1] : x'(0) = 1\}$	$\|x\| = \max_{0\le t\le \frac{1}{2}} \|x(t)\| + \max_{\frac{1}{2}\le t\le 1} \|x'(t)\|$
3.	$\{x(t) \in C[a,b] : x(a) = 0\}$	$\|x\| = \int_a^b \|x(t)\|dt$
4.	$\{x(t) \in C^{(1)}[a,b] : x'(t) < 0\}$	$\|x\| = (\int_a^b \|x(t)\|^2 dt)^{\frac{1}{2}}$
5.	$\{x \in l_\infty : \sum_{k=1}^{\infty} \|x(k)\|^2 < \infty\}$	$\|x\| = \sup_{k\in\mathbb{N}} \|x(k)\|$
6.	$\{x \in l_\infty : \sum_{k=1}^{\infty} \|x(k)\| < 1\}$	$\|x\| = \sum_{k=1}^{\infty} \|x(k)\|^2$
7.	$\{x(t) \in C[0,1] : x(0) = x(1)\}$	$\|x\| = \sup_{0\le t\le \frac{1}{2}} \|x(t)\|$
8.	$\{x(t) \in C[0,1] : x(0) = x(\frac{1}{2})\}$	$\|x\| = \sup_{0\le t\le \frac{1}{2}} \|x(t)\| + \int_{\frac{1}{2}}^1 \|x(t)\|dt$
9.	$\{x(t) \in C^{(1)}[a,b] : x'(a) = 0\}$	$\|x\| = \int_a^b \|x'(t)\|dt + \|x(a)\|$
10.	$\{x(t) \in C^{(1)}[a,b] : x(a) = x(b)\}$	$\|x\| = \int_a^{\frac{a+b}{2}} \|x(t)\|dt + \int_{\frac{a+b}{2}}^b \|x'(t)\|dt$

Definition 10. Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in X is called *a Cauchy sequence* if the terms of the sequence eventually all become arbitrarily close to one another, that is when the following condition hold

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n,m\ge N}\quad \|x_n - x_m\| < \varepsilon.$$

We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is *convergent in* X if there exists an element $x_0 \in X$ such that $\lim_{n\to\infty} \|x_n - x_0\| = 0$. Then we write $\lim_{n\to\infty} x_n = x_0$ and say that x_0 is the limit of x_n.

Ex 11. Show that the sequence $\{x_k\}_{k\in\mathbb{N}}$ in the space $(\mathbb{R}^n, \|\cdot\|_p)$, $p \in [1, \infty]$, is convergent to x_0 if and only if $\lim_{k\to\infty} x_k(i) = x_0(i)$ for all $i = 1, ..., n$.

Ex 12. Show that if a sequence $\{x_n\}_{n\in\mathbb{N}}$ in the space l_p, $p \in [1, \infty]$, is convergent to an element a then $\lim_{n\to\infty} x_n(k) = a(k)$ for all $k \in \mathbb{N}$. Is the converse implication true?

Ex 13. Check, whether a sequence x_n of elements of X converges to an element a.

N	X	x_n	a
1.	l_1	$(\underbrace{\sin\frac{1}{2^n}, \sin\frac{1}{2^n}, ..., \sin\frac{1}{2^n}}_{n}, 0, ...)$	$(0, 0, ..., 0, ...)$
2.	l_3	$(\underbrace{\frac{n^2}{2^n}, \frac{n^2}{2^n}, ..., \frac{n^2}{2^n}}_{n}, 0, ...)$	$(1, 0, ..., 0, ...)$
3.	c	$(\underbrace{(\frac{4n+1}{4n+3})^n, (\frac{4n+1}{4n+3})^n, ..., (\frac{4n+1}{4n+3})^n}_{n}, 0, ...)$	$(e^{-\frac{1}{2}}, e^{-\frac{1}{2}}, ..., e^{-\frac{1}{2}}, ...)$
4.	$l_{\frac{8}{5}}$	$(\underbrace{\frac{\cos\frac{1}{n}}{n}, \frac{\cos\frac{1}{n}}{n}, ..., \frac{\cos\frac{1}{n}}{n}}_{n}, 0, ...)$	$(0, 0, ..., 0, ...)$
5.	l_∞	$(0, \frac{7}{8}, ..., \frac{n^3-1}{n^3}, 0, 0, ...)$	$(0, \frac{7}{8}, ..., \frac{k^3-1}{k^3}, \frac{(k+1)^3-1}{(k+1)^3}, ...)$
6.	l_2	$(\underbrace{\frac{1}{n^2}, \frac{1}{n^2}, ..., \frac{1}{n^2}}_{n}, n, 0, ...)$	$(0, 0, ..., 0, ...)$

Ex 14. Examine convergence of a sequence x_n in a space X.

N	X	x_n	N	X	x_n
1.	l_∞	$(\underbrace{tg(1+\frac{1}{n})^n, tg(1+\frac{1}{n})^n, ..., tg(1+\frac{1}{n})^n}_{n}, 0, 0, ...)$	6.	$l_{\frac{5}{2}}$	$((\frac{n+1}{n})^n, (\frac{n+2}{n})^n, ..., (\frac{n+(n-1)}{n})^n, 0, 0, ...)$
2.	l_3	$(\underbrace{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}}_{n}, 1, 0, ...)$	7.	c	$(\frac{1}{2}, \frac{4}{5}, ..., \frac{n^2}{n^2+1}, 0, 0, ...)$
3.	l_2	$(\underbrace{\sin\frac{1}{n}, \sin\frac{1}{n}, ..., \sin\frac{1}{n}}_{n^2}, 0, 0, ...)$	8.	l_1	$(\underbrace{\frac{\sin 3^n}{n^2}, \frac{\sin 3^n}{n^2}, ..., \frac{\sin 3^n}{n^2}}_{n}, 0, 0, ...)$
4.	c_0	$(tg(\frac{1}{n}), tg(\frac{1}{n^2}), ..., tg(\frac{1}{n^k}), tg(\frac{1}{n^{k+1}}), ...)$	9.	l_∞	$(1, \sqrt[2]{2}, \sqrt[3]{3}..., \sqrt[n]{n}, 0, 0, ...)$
5.	l_2	$(\underbrace{0, .., 0}_{n}, \underbrace{\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}}_{n}, 0, 0, ...)$	10.	$l_{\sqrt{5}}$	$(\underbrace{\sin\frac{1}{\sqrt{n}}, \sin\frac{1}{\sqrt{n}}, ..., \sin\frac{1}{\sqrt{n}}}_{n}, 0, ...)$

Ex 15. Prove that the convergence in the space $C[a,b]$ (with norm $\|x\|_\infty = \sup_{t\in[a,b]}|x(t)|$) is equivalent to the uniform convergence of a sequence of functions.

Ex 16. Check, whether a sequence x_n of functions converges in $C[a,b]$ to α.

N	$C[a,b]$	x_n	α
1.	$C[-3,3]$	$\sqrt{t^2+\frac{1}{n^3}}$	$\vert t\vert$
2.	$C[0,8]$	$(\frac{t}{8})^n - (\frac{t}{8})^{2n} + t$	t
3.	$C[0,1]$	$t^{2n} - t^{n+1} + t$	t
4.	$C[-4,4]$	$\frac{1}{n^2}\sqrt{n^4t^2+1}$	t
5.	$C[1,2]$	$n^2(\sqrt{t+\frac{1}{n^3}} - \sqrt{t})$	$\frac{1}{2\sqrt{t}}$
6.	$C[\frac{1}{2},\frac{3}{2}]$	$\frac{t^n-t}{1+t^n}$	1
7.	$C[0,1]$	$\sqrt[n]{1+t^n}$	t
9.	$C[0,\frac{1}{3}]$	$3^nt^n - 3^{n+1}t^{n+1} - 3t^n$	0
10.	$C[0,2]$	$\sqrt[n]{1+t^n}$	$\begin{cases} 1, & t\in[0,1] \\ t, & t\in[1,2] \end{cases}$

Ex 17. Examine convergence of a sequence of functions x_n in a space X.

N	X	x_n
1.	$C[-1,0]$	$\frac{1}{n}\sqrt[3]{n^3t+1}$
2.	$C[1,2]$	$\frac{2t^n-1}{1+t^n}$
3.	$C[-1,\frac{1}{2}]$	$\frac{(t+1)^{2n}-t^2n}{(t+2)^{2n}}$
4.	$C[1,2]$	$n\sin(\frac{t^2}{n}) + \frac{t^3}{n}$
5.	$C[0,9]$	$\frac{9^nt^n-t^{2n}}{9^{2n}}$
6.	$C^{(1)}[0,1]$	$\frac{t^n}{n}$
7.	$C[-1,5]$	$\mathrm{arctg}(n(t^2+1))$
8.	$C[-\frac{\pi}{2},0]$	$(\sin t)^{2n} + \sqrt[3]{\frac{t}{n}}$
9.	$C[1,2]$	$\frac{t^2}{n^2}\ln(\frac{t}{n})$

Ex 18. Show that if a sequence is convergent it is a Cauchy sequence. Show by an example that the converse implication is not true.

Definition 19. A normed space $(X, \|\cdot\|)$ such that every Cauchy sequence is convergent in X is called a Banach space.

Ex 20. Prove that $(\mathbb{R}^n, \|\cdot\|_p)$, $p\in[1,\infty]$, are Banach spaces.